Module III - Macro-mechanics of Lamina

Lecture 23

Macro-Mechanics of Lamina

For better understanding of the macromechanics of lamina, the knowledge of the material properties in essential. Therefore, the type of materials and tests conducted to find material constants are discussed first.

Types of materials:

The materials are classified based on the behaviour for a particular loading condition. These include,

(i) Anisotropic materials
(ii) Monoclinic materials
(ii) Orthotropic materials
(iii) Transversely isotropic materials
(iv) Isotropic materials

Anisotropic materials:

In an anisotropic material, there are no planes of material property symmetry. So, it has different physical properties in different directions relative to the crystal orientation of the materials i.e., material properties are directionally dependent.

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{Bmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{pmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{Bmatrix} \qquad (3.1)$$

There are 21 independent elastic constants in the stress-strain relationship as given above. As shown in the above relations, there are couplings between the stresses and strains. Normal stresses produce not only normal strains in other directions due to Poisson effect but, also shear strains due to the effect of mutual influence. Similarly, shear stresses produce not only shear strains but also normal strains.

In an anisotropic material, a combination of extensional and shear deformation is produced by a normal stress acting in any direction. This phenomenon of creating both extensional and shear deformations by the application of either normal or shear stresses is termed as extension-shear coupling and is not observed in isotropic materials.

Monoclinic materials:

It has a single plane of material property symmetry. If xy plane (i.e.; 1-2 plane) is considered as the plane of material symmetry then, there are 13 independent elastic constants in the stiffness matrix as given below.

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{Bmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{55} & 0 \\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66} \end{pmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{Bmatrix} \quad (3.2)$$

As there is a single plane of material property symmetry, shear stresses from the planes in which one of the axis is the perpendicular axis of the plane of material symmetry (i.e.; 2-3 and 3-1 planes) will contribute only to the shear strains in those planes. And normal stresses will not contribute any shear strains in these planes.

Orthotropic materials:

There are three mutually orthogonal planes of material property symmetry in an orthotropic material. Fiber-reinforced composites, in general, contain the three orthogonal planes of material property symmetry and are classified as orthotropic materials. The intersections of these three planes of symmetry are called the principal material directions.

The material behaviour is called as specially orthotropic, when the normal stresses are applied in the principal material directions. Otherwise, it is called as general orthotropic which behaves almost equivalent to anisotropic material.

There are nine independent elastic constants in the stiffness matrix as given below for a specially orthotropic material.

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{Bmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{pmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{Bmatrix} \quad (3.3)$$

From the stress-strain relationship it is clear that normal stresses applied in one of the principal material directions on an orthotropic material cause elongation in the direction of the applied stresses and contractions in the other two transverse directions. However, normal stresses applied in any directions other than the principal material directions create both extensional and shear deformations.

Transversely isotropic materials:

If a material has axes of symmetry in its longitudinal axis and all directions perpendicular to its longitudinal axis (i.e., more than three mutually perpendicular axes of symmetry) then such material is transversely isotropic. (e.g., unidirectional composites). There are five independent elastic constants for these materials.

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{Bmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{(C_{11} - C_{12})}{2} \end{pmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{Bmatrix} \qquad (3.4)$$

Isotropic material:

In an isotropic material, properties are the same in all directions (axial, lateral, and in between). Thus, the material contains an infinite number of planes of material property symmetry passing through a point. i.e., material properties are directionally independent. So, there are two independent elastic constants.

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{Bmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{(C_{11} - C_{12})}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{(C_{11} - C_{12})}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{(C_{11} - C_{12})}{2} \end{pmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{Bmatrix} \qquad (3.5)$$

Tensile normal stresses applied in any direction on an isotropic material cause only elongation in the direction of the applied stresses and contractions in the two transverse directions. It will not produce any shear strain in any form in the material. Similarly, shear stresses produce only corresponding shear strains not normal strains.

As the material properties are directionly independent, isotropic material has equal strength in all directions. As such, efficient structure (without wasting material structurally) is not possible by isotropic material (e.g., beam). In a beam the load is applied on the transverse direction and the beam will bend by extending and contracting in the lengthwise direction. In the lateral direction there is no load applied and the strength in the lateral direction is under utilized. Hence, isotropic material in beam structure is not efficient design, as the material is under utilized. On the other hand, unidirectional fibers which are aligned in the lengthwise direction will have high strength in the lengthwise direction and low strength in the other direction and thus, it becomes an efficient design. If the structure is taking complex loading system then, isotropic properties will be ideal and by arranging the fibers randomly quasi isotropic properties can be achieved.

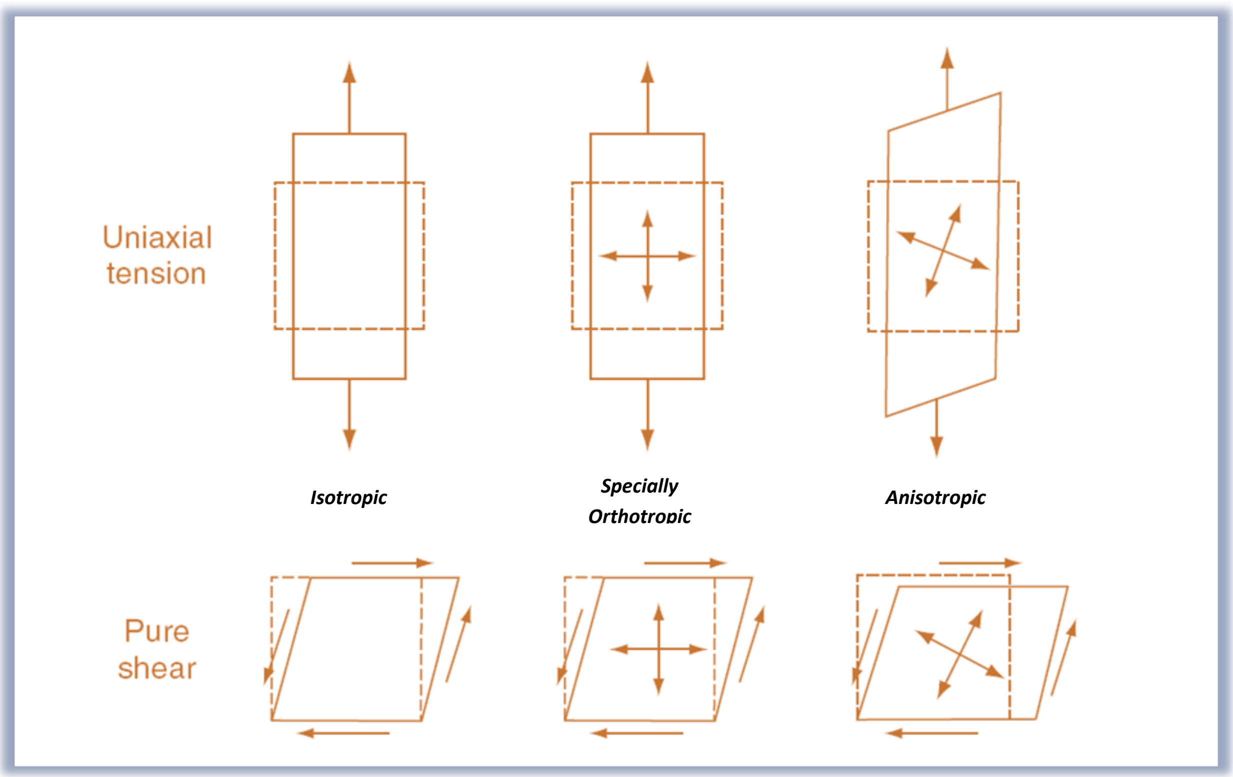


Figure 3.1 Differences in the deformations on uniaxial tension and pure shear

From figure 3.1, it is clear that isotropic and specially orthotropic materials behave in a similar way. But, the magnitude of deformation is direction dependent in the case of orthotropic whereas in the case of isotropic it is not. Anisotropic (and generally, orthotropic material) has coupling between normal and shear deformations.

To summarize the elastic constants of each type of material is given in Table 1.27

Table 1.27: Elastic constants of different materials

Material	**Three dimensional**		**Two dimensional**	
	Number of non zero constants	Number of independent constants	Number of non zero constants	Number of independent constants
Anisotropic	36	21	9	6
Generally Orthotropic	36	9	9	4
Specially Orthotropic	12	9	5	4
Transversely Isotropic	12	5	5	4
Isotropic	12	2	5	2

References :

1) Analysis and Performance of Fibre composites, - B.D. Agarwal, L.J. Broughtman and K.Chandrashekar John Wiley & Sons. Inc.

2) Principles of Composite Material Mechanics, R.F. Gibson, CRC Press.

Lecture 24

Two-Dimensional Unidirectional Lamina:

Consider, a thin plate of unidirectional composite material of uniform cross-section. If there are loads applied only along the edges and no out-of-plane loads, then, it can be considered to be a plane stress case.

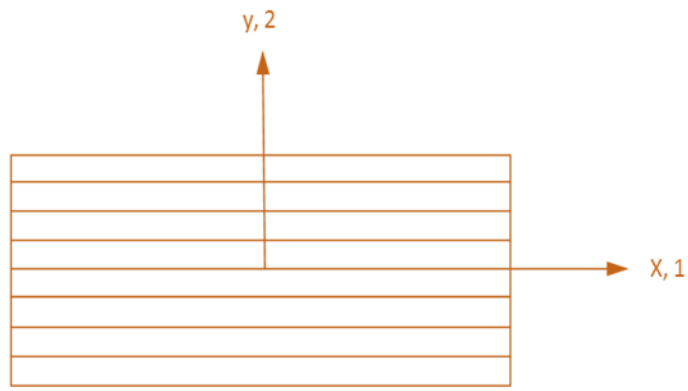


Figure 3.2 Orthotropic lamina

Since, the plate is very thin, the stresses normal to plane of loading i.e. σ_z , τ_{xz} and τ_{yz} can be assumed to vary insignificantly across the thickness. Thus, they can be assumed to be zero within the plate. i.e.

$$\sigma_z = 0,\ \tau_{xz} = 0,\ \text{and}\ \tau_{yz} = 0 \quad (3.6)$$

This assumption reduces the three-dimensional stress–strain equations to two-dimensional stress–strain equations.

From generalized Hooke's law,

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right] \quad (3.7)$$

If $\sigma_z = 0$, then

$$\varepsilon_z = -\frac{\nu}{E}\left(\sigma_x + \sigma_y\right) \quad (3.8)$$

Let, the principal material directions be designated by the longitudinal direction L and the transverse direction T. Considering various loading conditions, the corresponding strains in the longitudinal and transverse directions are determined as follows.

Case (i) Only longitudinal load is applied

Therefore,

$$\sigma_L \neq 0 \quad , \quad \sigma_T = \tau_{LT} = 0 \qquad (3.9)$$

Figure 3.3 Longitudinal loading condition

The strains corresponding to this loading condition are,

$$\varepsilon_L = \frac{\sigma_L}{E_L} \qquad (3.10)$$

$$\varepsilon_T = -\nu_{\mathrm{LT}} \frac{\sigma_L}{E_L} \qquad (3.11)$$

$$\gamma_{LT} = 0 \qquad (3.12)$$

Case (ii) Only transverse load is applied

Therefore,

$$\sigma_T \neq 0 \; , \; \sigma_L = \tau_{LT} = 0$$

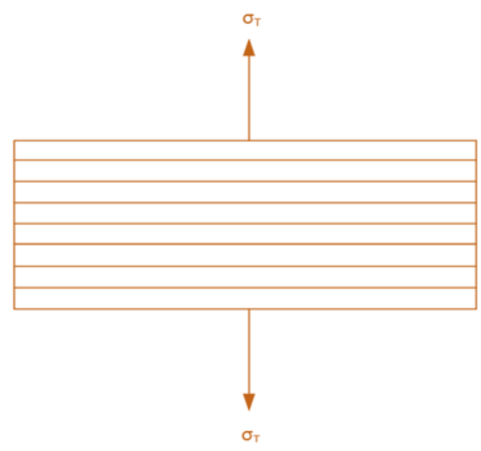


Figure 3.4 Transverse loading condition

The strains corresponding to this loading condition are,

$$\varepsilon_T = \frac{\sigma_T}{E_T} \qquad (3.13)$$

$$\varepsilon_L = -\nu_{TL}\frac{\sigma_T}{E_T} \qquad (3.14)$$

$$\gamma_{LT} = 0$$

Case (iii) Only shear load is applied

Therefore,

$$\tau_{LT} \neq 0\ ,\ \sigma_L = \sigma_T = 0$$

Figure 3.5 Shear loading condition

The strains corresponding to this loading condition are,

$$\gamma_{LT} = \frac{\tau_{LT}}{G_{LT}} \qquad (3.15)$$

$$\varepsilon_L = \varepsilon_T = 0 \qquad (3.16)$$

Case (iv) All loads are applied

Superimposing cases (i), (ii) and (iii)

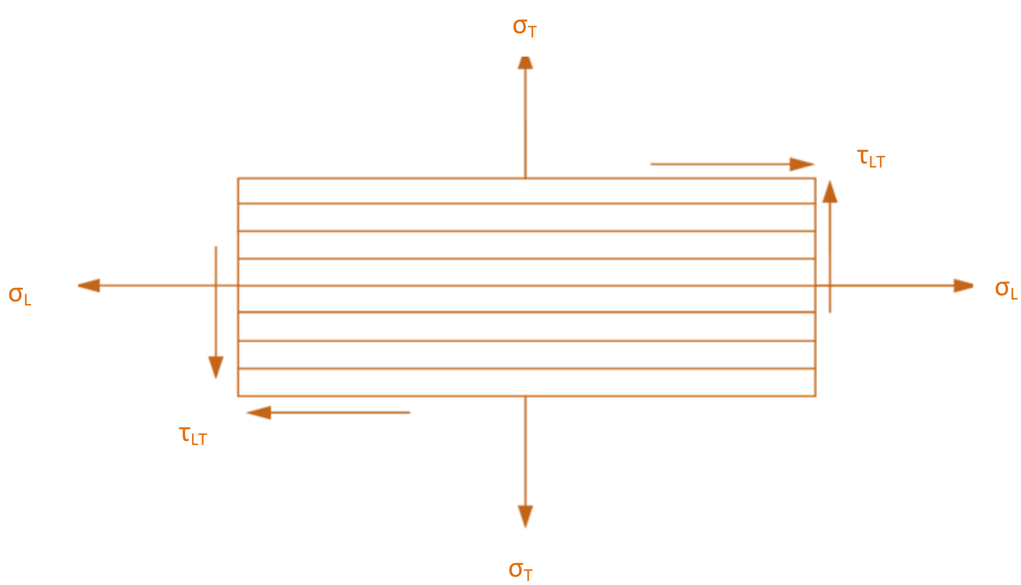


Figure 3.6 Lamina under loading condition

The strains corresponding to this loading condition are,

$$\varepsilon_L = \frac{\sigma_L}{E_L} - \nu_{TL}\frac{\sigma_T}{E_T} \quad (3.17)$$

$$\varepsilon_T = \frac{\sigma_T}{E_T} - \nu_{LT}\frac{\sigma_L}{E_L} \quad (3.18)$$

$$\gamma_{LT} = \frac{\tau_{LT}}{G_{LT}}$$

Anisotropic layer (Generally Orthotropic):

When an orthotropic lamina is loaded in the direction other than its principal material axes, then, the behaviour of the lamina will be anisotropic. In the figure 3.7, the lamina will behave like anisotropic if load is applied along x or y directions.

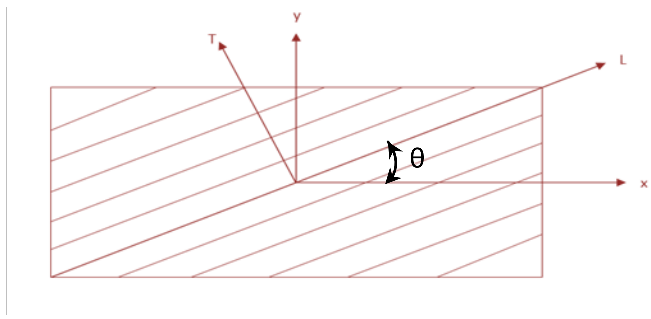


Figure 3.7 Generally Orthotropic lamina

<u>Case (i) Only longitudinal load is applied (along x direction)</u>

Therefore, $\sigma_x \neq 0,$

$\sigma_y = \tau_{xy} = 0$

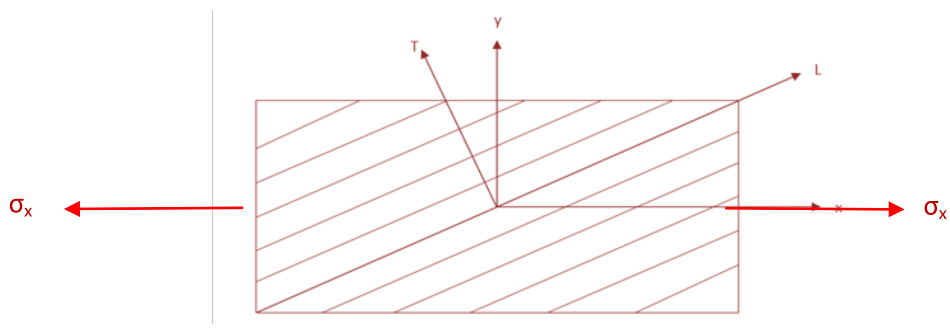


Figure 3.8 Longitudinal loading condition

The strains corresponding to this loading condition are,

$$\varepsilon_x = \frac{\sigma_x}{E_x} \quad (3.19)$$

$$\varepsilon_y = -\nu_{\mathbf{xy}} \frac{\sigma_x}{E_x} \quad (3.20)$$

$$\gamma_{xy} = -m_x \frac{\sigma_x}{E_L} \quad (3.21)$$

where, m_x is the coefficient of mutual influence and

E_L is the modulus of composite along L direction

Case (ii) Only transverse load is applied

Therefore, $\sigma_y \neq \mathbf{0}$

$\sigma_x = \tau_{xy} = \mathbf{0}$

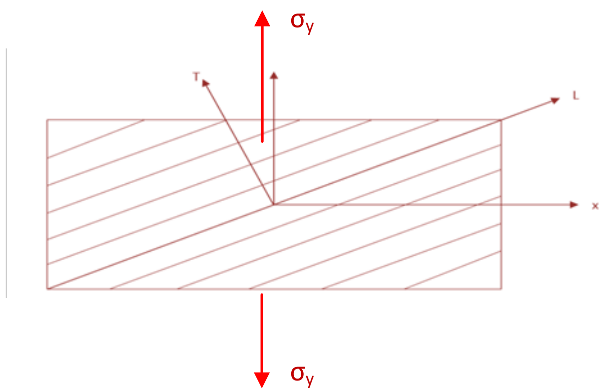


Figure 3.9 Transverse loading condition

The strains corresponding to this loading condition are,

$$\varepsilon_y = \frac{\sigma_y}{E_y} \quad (3.22)$$

$$\varepsilon_x = -\nu_{\mathbf{yx}} \frac{\sigma_y}{E_y} \quad (3.23)$$

$$\gamma_{xy} = -m_y \frac{\sigma_y}{E_L} \quad (3.24)$$

Case (ii) Only shear load is applied

Therefore, $\tau_{xy} \neq \mathbf{0}$

$\sigma_x = \sigma_y = \mathbf{0}$

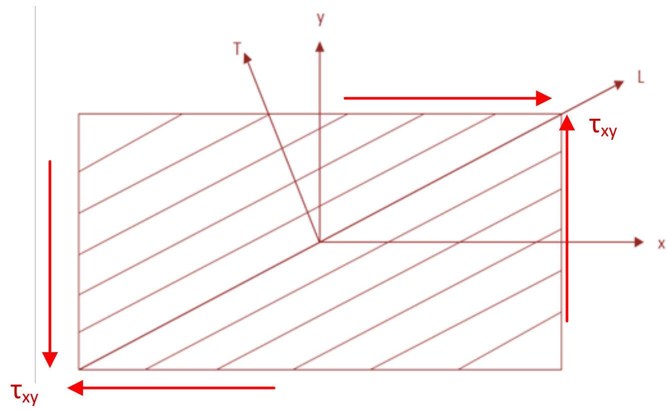


Figure 3.10 Shear loading condition

The strains corresponding to this loading condition are,

$$\varepsilon_x = -m_x \frac{\tau_{xy}}{E_L} \qquad (3.25)$$

$$\varepsilon_y = -m_y \frac{\tau_{xy}}{E_L} \qquad (3.26)$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G_{xy}} \qquad (3.27)$$

Case (iv) All loads are applied

Superimposing cases (i), (ii) and (iii)

$$\varepsilon_x = \frac{\sigma_x}{E_x} - \nu_{yx}\frac{\sigma_y}{E_y} - m_x \frac{\tau_{xy}}{E_L} \qquad (3.28)$$

$$\varepsilon_y = \frac{\sigma_y}{E_y} - \nu_{xy}\frac{\sigma_x}{E_x} - m_y \frac{\tau_{xy}}{E_L} \qquad (3.29)$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G_{xy}} - m_x \frac{\sigma_x}{E_L} - m_y \frac{\sigma_y}{E_L} \qquad (3.30)$$

References :

1) Analysis and Performance of Fibre composites, - B.D. Agarwal, L.J. Broughtman and K.Chandrashekar John Wiley & Sons. Inc.

2) Principles of Composite Material Mechanics, R.F. Gibson, CRC Press.

Problems :

Calculate E_x, E_y, γ_{xy} , M_x and M_y at 45^o draw orthotropic lamina of E_L = 35 GPa , E_T = 3.5 GPa , G_{LT} = 4 GPa and γ_{CT} = 0.4 .

Find, $\in_x , \in_y , \in_{xy}$ when, σ_x = 20 MPa .

Lecture 25

Transformation of Engineering constants:

It is of interest to derive and know the explicit expressions for the usual engineering constants in the arbitrary axes in terms of those, along the principal material axes.

Consider an orthotropic lamina with its principal material axes (L and T) oriented at an angle θ with reference axes (x and y) as shown in Fig. 3.11

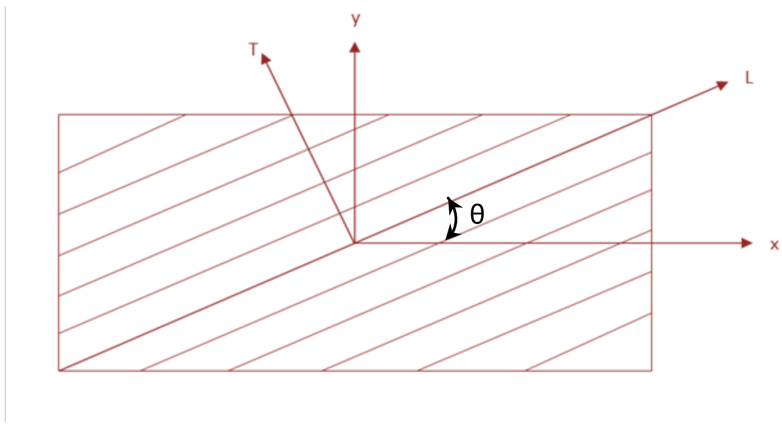


Fig. 3.11 Orthotropic lamina

Let, the only nonzero stress acting on this lamina be σ_x. i.e.,

$\sigma_x \neq 0$

$\sigma_y = \tau_{xy} = 0$

The normal and shearing stresses along the L and T directions can be calculated by the stress-transformation law:

$$\sigma_L = \sigma_x \cos^2\theta + \sigma_y \sin 2\theta + 2\tau_{Xy} \sin\theta \cos\theta$$

$$\therefore \sigma_L = \sigma_x \cos^2\theta \qquad (3.31)$$

$$\sigma_T = \sigma_x \sin^2\theta + \sigma_y \cos 2\theta - 2\tau_{Xy} \sin\theta \cos\theta$$

$$\therefore \sigma_T = \sigma_x \sin^2\theta \qquad (3.32)$$

$$\tau_{LT} = -\sigma_x \sin\theta\cos\theta + \sigma_y \sin\theta\cos\theta + \tau_{xy}\left(\cos 2\theta - \sin 2\theta\right)$$

$$\therefore \tau_{LT} = -\sigma_x \sin\theta\cos\theta \qquad (3.33)$$

The above relations are substituted in the strain relations derived previously for the orthotropic lamina in the L and T directions.

$$\varepsilon_L = \frac{\sigma_L}{E_L} - \nu_{\mathbf{TL}} \frac{\sigma_T}{E_T}$$

$$\varepsilon_L = \frac{\sigma_x \cos^2\theta}{E_L} - \nu_{\mathbf{TL}} \frac{\sigma_x \sin^2\theta}{E_T} \qquad (3.34)$$

$$\varepsilon_T = \frac{\sigma_T}{E_T} - \nu_{\mathbf{LT}} \frac{\sigma_L}{E_L}$$

$$\varepsilon_T = \frac{\sigma_x \sin^2\theta}{E_T} - \nu_{\mathbf{LT}} \frac{\sigma_x \cos^2\theta}{E_L} \qquad (3.35)$$

$$\gamma_{LT} = \frac{\tau_{LT}}{G_{LT}}$$

$$\gamma_{LT} = -\frac{\sigma_x \sin\theta \cos\theta}{G_{LT}} \qquad (3.36)$$

The strains in the x and y directions can be obtained as by taking the inverse of the strain-transformation law, which can be written as :

$$\varepsilon_x = \varepsilon_L \cos^2\theta + \varepsilon_T \sin^2\theta - \gamma_{LT} \sin\theta\cos\theta \qquad (3.37)$$

$$\varepsilon_y = \varepsilon_L \sin^2\theta + \varepsilon_T \cos^2\theta + \gamma_{LT} \sin\theta\cos\theta \qquad (3.38)$$

$$\gamma_{xy} = 2(\varepsilon_L - \varepsilon_T)\sin\theta\cos\theta + \gamma_{LT}(\cos^2\theta - \sin^2\theta) \qquad (3.39)$$

Substitution of the values of ε_L, ε_T, and γ_{LT} the above equations become,

$$\varepsilon_x = \left(\frac{\sigma_x \cos^2\theta}{E_L} - \nu_{\mathbf{TL}} \frac{\sigma_x \sin^2\theta}{E_T}\right)\cos^2\theta + \left(\frac{\sigma_x \sin^2\theta}{E_T} - \nu_{\mathbf{LT}} \frac{\sigma_x \cos^2\theta}{E_L}\right)\sin^2\theta + \left(\frac{\sigma_x \sin\theta\cos\theta}{G_{LT}}\right)\sin\theta\cos\theta$$

$$= \sigma_x\left[\left(\frac{\cos^2\theta}{E_L} - \nu_{\mathbf{TL}} \frac{\sin^2\theta}{E_T}\right)\cos^2\theta + \left(\frac{\sin^2\theta}{E_T} - \nu_{\mathbf{LT}} \frac{\cos^2\theta}{E_L}\right)\sin^2\theta + \left(\frac{\sin\theta\cos\theta}{G_{LT}}\right)\sin\theta\cos\theta\right]$$

$$= \sigma_x\left[\frac{cos^4\,\theta}{E_L} - \nu_{TL}\frac{sin^2\,\theta\,cos^2\,\theta}{E_T} + \frac{sin^4\,\theta}{E_T} - \nu_{LT}\frac{sin^2\,\theta\,cos^2\,\theta}{E_L} + \frac{sin^2\,\theta\,cos^2\,\theta}{G_{LT}}\right]$$

$$= \sigma_x\left[\frac{cos^4\,\theta}{E_L} - \nu_{LT}\frac{sin^2\,\theta\,cos^2\,\theta}{E_L} + \frac{sin^4\,\theta}{E_T} - \nu_{LT}\frac{sin^2\,\theta\,cos^2\,\theta}{E_L} + \frac{sin^2\,\theta\,cos^2\,\theta}{G_{LT}}\right]$$

$$= \sigma_x\left[\frac{cos^4\,\theta}{E_L} + \frac{sin^4\,\theta}{E_T} - 2\nu_{LT}\frac{sin^2\,\theta\,cos^2\,\theta}{E_L} + \frac{sin^2\,\theta\,cos^2\,\theta}{G_{LT}}\right]$$

$$\boldsymbol{\varepsilon_x = \sigma_x\left[\frac{cos^4\,\theta}{E_L} + \frac{sin^4\,\theta}{E_T} + \frac{1}{4}\left(\frac{1}{G_{LT}} - \frac{2\nu_{LT}}{E_L}\right)sin^2\,2\theta\right]} \qquad (3.40)$$

$$\varepsilon_y = \varepsilon_L\,sin^2\,\theta + \varepsilon_T\,cos^2\,\theta + \gamma_{LT}\,sin\theta\,cos\theta \qquad (3.41)$$

$$= \left(\frac{\sigma_x\,cos^2\,\theta}{E_L} - \nu_{TL}\frac{\sigma_x\,sin^2\,\theta}{E_T}\right)\sin^2\theta + \left(\frac{\sigma_x\,sin^2\,\theta}{E_T} - \nu_{LT}\frac{\sigma_x\,cos^2\,\theta}{E_L}\right)\cos^2\theta - \left(\frac{\sigma_x\,sin\theta\,cos\theta}{G_{LT}}\right)sin\theta cos\theta$$

$$= \sigma_x\left[\frac{sin^2\,\theta\,cos^2\,\theta}{E_L} - \nu_{TL}\frac{sin^4\,\theta}{E_T} + \frac{sin^2\,\theta\,cos^2\,\theta}{E_T} - \nu_{LT}\frac{cos^4\theta}{E_L} - \frac{sin^2\,\theta\,cos^2\theta}{G_{LT}}\right]$$

$$= \sigma_x\left[\left(\frac{1}{E_L} + \frac{1}{E_T} - \frac{1}{G_{LT}}\right)sin^2\,\theta\,cos^2\,\theta - (sin^4\,\theta + cos^4\,\theta)\frac{\nu_{LT}}{E_L}\right]$$

From trigonometry:

$$(sin^2\,\theta + cos^2\,\theta)^2 = sin^4\,\theta + cos^4\,\theta + 2\,sin^2\,\theta\,cos^2\,\theta$$

$$sin^4\,\theta + cos^4\,\theta = 1 - 2\,sin^2\,\theta\,cos^2\,\theta \qquad (3.42)$$

$$= \sigma_x\left[\left(\frac{1}{E_L} + \frac{1}{E_T} - \frac{1}{G_{LT}}\right)sin^2\,\theta\,cos^2\,\theta - (1 - 2\,sin^2\,\theta\,cos^2\,\theta)\frac{\nu_{LT}}{E_L}\right]$$

$$= -\sigma_x \left[\frac{\nu_{LT}}{E_L} - \left(\frac{1}{E_L} + \frac{1}{E_T} + \frac{2\nu_{LT}}{E_L} - \frac{1}{G_{LT}} \right) \sin^2\theta \cos^2\theta \right]$$

$$\varepsilon_y = -\sigma_x \left[\frac{\nu_{LT}}{E_L} - \frac{1}{4} \left(\frac{1}{E_L} + \frac{1}{E_T} + \frac{2\nu_{LT}}{E_L} - \frac{1}{G_{LT}} \right) \sin^2 2\theta \right] \quad (3.43)$$

$$\gamma_{xy} = 2(\varepsilon_L - \varepsilon_T)\sin\theta\cos\theta + \gamma_{LT}(\cos^2\theta - \sin^2\theta) \quad (3.44)$$

$$\gamma_{xy} = 2\left(\frac{\sigma_x \cos^2\theta}{E_L} - \nu_{TL}\frac{\sigma_x \sin^2\theta}{E_T} - \frac{\sigma_x \sin^2\theta}{E_T} + \nu_{LT}\frac{\sigma_x \cos^2\theta}{E_L} \right)\sin\theta\cos\theta - \frac{\sigma_x \sin\theta\cos\theta}{G_{LT}}(\cos^2\theta - \sin^2\theta)$$

$$= \sigma_x \left(\frac{2\cos^3\theta\sin\theta}{E_L} - \nu_{TL}\frac{2\sin^3\theta\cos\theta}{E_T} - \frac{2\sin^3\theta\cos\theta}{E_T} + \nu_{LT}\frac{2\cos^3\theta\sin\theta}{E_L} - \frac{\cos^3\theta\sin\theta}{G_{LT}} + \frac{\sin^3\theta\cos\theta}{G_{LT}} \right)$$

$$= \sigma_x \sin 2\theta \left(\frac{\cos^2\theta}{E_L} - \nu_{TL}\frac{\sin^2\theta}{E_T} - \frac{\sin^2\theta}{E_T} + \nu_{LT}\frac{\cos^2\theta}{E_L} - \frac{\cos^2\theta}{2G_{LT}} + \frac{\sin^2\theta}{2G_{LT}} \right)$$

$$\gamma_{xy} = \sigma_x \sin 2\theta \left[-\frac{\nu_{LT}}{E_L} - \frac{1}{E_T} + \frac{1}{2G_{LT}} + \cos^2\theta \left(\frac{1}{E_L} + \frac{2\nu_{LT}}{E_L} + \frac{1}{E_T} - \frac{1}{G_{LT}} \right) \right] \quad (3.45)$$

To summarize,

$$\epsilon_x = \sigma_x \left[\frac{\cos^4\theta}{E_L} + \frac{\sin^4\theta}{E_T} + \frac{1}{4}\left(\frac{1}{G_{LT}} - \frac{2\nu_{LT}}{E_L} \right)\sin^2 2\theta \right] \quad (3.46)$$

$$\epsilon_y = -\sigma_x \left[\frac{\nu_{LT}}{E_L} - \frac{1}{4}\left(\frac{1}{E_L} + \frac{1}{E_T} + \frac{2\nu_{LT}}{E_L} - \frac{1}{G_{LT}} \right)\sin^2 2\theta \right] \quad (3.47)$$

$$\gamma_{xy} = \sigma_x \sin 2\theta \left[-\frac{\nu_{LT}}{E_L} - \frac{1}{E_T} + \frac{1}{2G_{LT}} + \cos^2\theta \left(\frac{1}{E_L} + \frac{2\nu_{LT}}{E_L} + \frac{1}{E_T} - \frac{1}{G_{LT}} \right) \right] \quad (3.47)$$

Determinations of Elastic constants

Once the strains are found, the elastic constants can be determined from the Hooke's law. The modulus of elasticity in the x-direction is given by :

$$E_x = \frac{\sigma_x}{\epsilon_x} \tag{3.48}$$

From the previous equation for ε_x, E_x can be calculated as :

$$\frac{1}{E_x} = \frac{\cos^4\theta}{E_L} + \frac{\sin^4\theta}{E_T} + \frac{1}{4}\left(\frac{1}{G_{LT}} - \frac{2\nu_{LT}}{E_L}\right)\sin^2 2\theta \tag{3.49}$$

The expression for E_y can be obtained by substituting θ + 90º for θ in above equation :

$$\frac{1}{E_y} = \frac{\sin^4\theta}{E_L} + \frac{\cos^4\theta}{E_T} + \frac{1}{4}\left(\frac{1}{G_{LT}} - \frac{2\nu_{LT}}{E_L}\right)\sin^2 2\theta \tag{3.50}$$

The Poisson ratio is defined as :

$$\nu_{xy} = -\frac{\varepsilon_y}{\varepsilon_x}$$

$$\nu_{xy} = -\varepsilon_y\left(\frac{E_x}{\sigma_x}\right) \tag{3.51}$$

$$\nu_{xy} = \left(\frac{E_x}{\sigma_x}\right)\sigma_x\left[\frac{\nu_{LT}}{E_L} - \frac{1}{4}\left(\frac{1}{E_L} + \frac{1}{E_T} + \frac{2\nu_{LT}}{E_L} - \frac{1}{G_{LT}}\right)\sin^2 2\theta\right] \tag{3.52}$$

$$\frac{\nu_{xy}}{E_x} = \frac{\nu_{LT}}{E_L} - \frac{1}{4}\left(\frac{1}{E_L} + \frac{1}{E_T} + \frac{2\nu_{LT}}{E_L} - \frac{1}{G_{LT}}\right)\sin^2 2\theta \tag{3.53}$$

Similarly,

$$\frac{\nu_{yx}}{E_y} = \frac{\nu_{TL}}{E_T} - \frac{1}{4}\left(\frac{1}{E_L} + \frac{1}{E_T} + \frac{2\nu_{LT}}{E_L} - \frac{1}{G_{LT}}\right)\sin^2 2\theta \tag{3.54}$$

When the normal stress σ_x is applied in a direction other than the L and T direction, it may induce a shearing strain given by the above equations. Therefore, a coefficient of mutual influence, m_x, may be defined that relates the shearing strain to the normal stress σ_x in the following manner:

$$\gamma_{xy} = -m_x\frac{\sigma_x}{E_L} \tag{3.55}$$

$$m_x = -\gamma_{xy}\frac{E_L}{\sigma_x} \tag{3.56}$$

Substituting the values σ_x and γ_{xy},

$$m_x = \sin 2\theta\left[\nu_{LT} + \frac{E_L}{E_T} - \frac{E_L}{2G_{LT}} - \cos^2\theta\left(1 + 2\nu_{LT} + \frac{E_L}{E_T} - \frac{E_L}{G_{LT}}\right)\right] \qquad (3.57)$$

Similarly, the coefficient, m_y, which relates the shearing strain to normal stress σ_y is defined as :

$$\gamma_{xy} = -m_y \frac{\sigma_y}{E_L} \qquad (3.58)$$

$$m_y = \sin 2\theta\left[\nu_{LT} + \frac{E_L}{E_T} - \frac{E_L}{2G_{LT}} - \sin^2\theta\left(1 + 2\nu_{LT} + \frac{E_L}{E_T} - \frac{E_L}{G_{LT}}\right)\right] \qquad (3.59)$$

To obtain an expression for G_{xy}, assume that the only non-zero stress acting on the lamina is τ_{xy} (Pure shear case). The stresses along the principal material directions are given by :

$$\sigma_L = 2\tau_{xy}\sin\theta\cos\theta \qquad (3.60)$$

$$\sigma_T = -2\tau_{xy}\sin\theta\cos\theta \qquad (3.61)$$

$$\tau_{LT} = \tau_{xy}(\cos^2\theta - \sin^2\theta) \qquad (3.62)$$

The corresponding strains are given by Hooke's law,

$$\varepsilon_L = \frac{\sigma_L}{E_L} - \nu_{TL}\frac{\sigma_T}{E_T}$$

$$\varepsilon_L = \left(\frac{1}{E_L} + \frac{\nu_{TL}}{E_T}\right) 2\tau_{xy}\sin\theta\cos\theta \qquad (3.63)$$

$$\varepsilon_T = \frac{\sigma_T}{E_T} - \nu_{LT}\frac{\sigma_L}{E_L}$$

$$\varepsilon_T = \left(\frac{1}{E_T} + \frac{\nu_{LT}}{E_L}\right) - 2\tau_{xy}\sin\theta\cos\theta \qquad (3.64)$$

$$\gamma_{LT} = \frac{\tau_{LT}}{G_{LT}}$$

$$\gamma_{LT} = \frac{\tau_{xy}}{G_{LT}}(\cos^2\theta - \sin^2\theta) \qquad (3.65)$$

Substitution of the above equations gives the shearing strain γ_{xy},

$$\gamma_{xy} = 2(\epsilon_L - \epsilon_T)\sin\theta\cos\theta + \gamma_{LT}(\cos^2\theta - \sin^2\theta)$$

$$\gamma_{xy} = 2\left[\left(\frac{1}{E_L} + \frac{\nu_{TL}}{E_T}\right) 2\tau_{xy} \sin\theta \cos\theta - \left(\frac{1}{E_T} + \frac{\nu_{LT}}{E_L}\right) - 2\tau_{xy} \sin\theta \cos\theta\right] \sin\theta \cos\theta + \frac{\tau_{xy}}{G_{LT}} (\cos^2\theta - \sin^2\theta)(\cos^2\theta - \sin^2\theta) \quad (3.66)$$

$$\gamma_{xy} = 4\tau_{xy}\left\{\left[\left(\frac{1}{E_L} + \frac{\nu_{TL}}{E_T}\right) - \left(\frac{1}{E_T} + \frac{\nu_{LT}}{E_L}\right) - 1\right] \sin 2\theta \cos 2\theta + \frac{1}{G_{LT}} (\cos^2\theta - \sin^2\theta)2\right\}$$

$$\frac{1}{G_{xy}} = \frac{\gamma xy}{\tau xy} = \frac{1}{E_L} + \frac{2\nu_{LT}}{E_L} + \frac{1}{E_T} - \left(\frac{1}{E_L} + \frac{2\nu_{LT}}{E_L} + \frac{1}{E_T} - \frac{1}{G_{LT}}\right)\cos^2\theta \quad (3.67)$$

As the normal stresses do, the shearing stress τ_{xy} will also cause direct strains ε_x, and ε_y in the x and y directions, respectively, given by :

$$\varepsilon_x = -m_x \frac{\tau_{xy}}{E_L} \quad (3.68)$$

$$\varepsilon_y = -m_y \frac{\tau_{xy}}{E_L} \quad (3.69)$$

It will be relevant at this point to note that the stress-strain relations for an orthotropic lamina referred to arbitrary axes can be written in terms of engineering constants as :

$$\varepsilon_x = \frac{\sigma_x}{E_x} - \nu_{yx}\frac{\sigma_y}{E_y} - m_x \frac{\tau_{xy}}{E_L} \quad (3.70)$$

$$\varepsilon_y = \frac{\sigma_y}{E_y} - \nu_{xy}\frac{\sigma_x}{E_x} - m_y \frac{\tau_{xy}}{E_L} \quad (3.71)$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G_{xy}} - m_x \frac{\sigma_x}{E_L} - m_y \frac{\sigma_y}{E_L} \quad (3.72)$$

The elastic constants E_x, E_y, ν_{xy}, ν_{yx}, m_x, m_y, and G_{xy} are determined from the equations given above.

Lecture 26

Specially Orthotropic material under Plane stress:

In plane stress condition, the stresses normal to the plane under consideration are assumed to be zero. i.e., if the 1-2 (xy) plane is assumed to be the loading plane, then the normal stresses to 1-2 plane

$$\sigma_3 = \tau_{23} = \tau_{13} = 0 \tag{3.73}$$

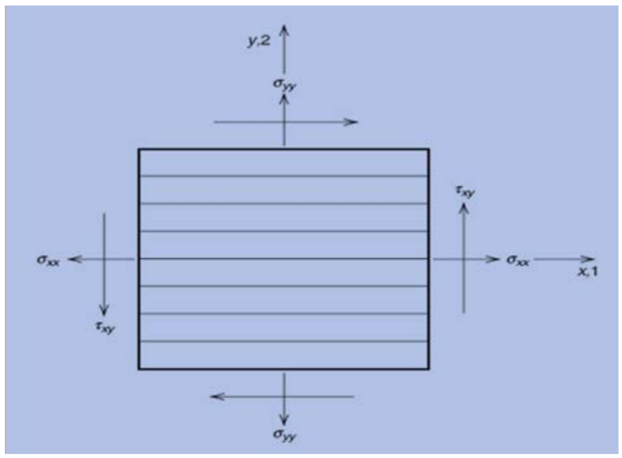


Fig. 3.12 Orthotropic lamina

More over, the number of independent elastic constants for orthotropic material in the plane stress condition are reduced to 4 from 9 in the three-dimensional case. Thus, the stress-strain relation is given by

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33}=0 \\ \tau_{23}=0 \\ \tau_{31}=0 \\ \tau_{12} \end{Bmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{pmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{Bmatrix} \tag{3.74}$$

$$\sigma_{11} = C_{11}\,\varepsilon_{11} + C_{12}\,\varepsilon_{22} + C_{13}\,\varepsilon_{33} \tag{3.75}$$

$$\sigma_{22} = C_{12}\ \varepsilon_{11} + C_{22}\ \varepsilon_{22} + C_{23}\ \varepsilon_{33} \qquad (3.76)$$

$$0 = C_{13}\ \varepsilon_{11} + C_{32}\ \varepsilon_{22} + C_{33}\ \varepsilon_{33}$$

$$\gamma_{23} = 0$$

$$\gamma_{13} = 0$$

$$\tau_{12} = C_{66}\ \gamma_{12}$$

after eliminating ε_{33}, the equations may be written as

$$\sigma_{11}\ or\ \sigma_1 = \left(c_{11} - \frac{C_{132}}{C_{33}}\right)\varepsilon_1 + \left(c_{12} - \frac{C_{13}C_{23}}{C_{33}}\right)\varepsilon_2 \qquad (3.77)$$

$$\sigma_{22}\ or\ \sigma_2 = \left(c_{12} - \frac{c_{13}\,c_{23}}{c_{33}}\right)\varepsilon_1 + \left(c_{22} - \frac{C_{232}}{C_{33}}\right)\varepsilon_2 \qquad (3.78)$$

$$\tau_{12} = C_{66}\ \gamma_{12} \qquad (3.79)$$

or,

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} \qquad (3.80)$$

where,

$$Q_{11} = \left(c_{11} - \frac{C_{132}}{C_{33}}\right) \qquad (3.81)$$

$$Q_{22} = \left(c_{22} - \frac{C_{232}}{C_{33}}\right) \qquad (3.82)$$

$$Q_{12} = \left(c_{12} - \frac{C_{13}\,C_{23}}{C_{33}}\right) \qquad (3.83)$$

$$Q_{66} = G_{12}$$

for specially orthotropic composite materials, the stiffness coefficients may be related with engineering constants as follows:

$$Q_{11} = \frac{E_L}{1-\nu_{LT}\,\nu_{TL}} \quad (3.84)$$

$$Q_{22} = \frac{E_T}{1-\nu_{LT}\,\nu_{TL}} \quad (3.85)$$

$$Q_{12} = \frac{\nu_{LT}\,E_T}{1-\nu_{LT}\,\nu_{TL}} = \frac{\nu_{TL}\,E_L}{1-\nu_{LT}\,\nu_{TL}} \quad (3.86)$$

$$Q_{66} = G_{LT} \quad (3.87)$$

The strain-stress relationship is given by,

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} \quad (3.88)$$

where

$$S_{11} = \frac{1}{E_{11}} \quad (3.89)$$

$$S_{22} = \frac{1}{E_{22}} \quad (3.90)$$

$$S_{12} = -\frac{\nu_{12}}{E_{11}} = -\frac{\nu_{21}}{E_{22}} \quad (3.91)$$

$$S_{66} = \frac{1}{G_{12}} \quad (3.92)$$

Stress-strain relations for thin isotropic lamina:

Stresses in an isotropic lamina under a plane stress condition is given by

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} \quad (3.93)$$

where

$$Q_{11} = Q_{22} = \frac{E}{1-\nu^2} \quad (3.94)$$

$$Q_{12} = \frac{\nu E}{1-\nu^2} \quad (3.95)$$

$$Q_{66} = G = \frac{E}{2(1+\nu)} \quad (3.96)$$

and the strain-stress relation is given by

$$\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{Bmatrix}=\begin{bmatrix}S_{11} & S_{12} & 0\\ S_{12} & S_{22} & 0\\ 0 & 0 & S_{66}\end{bmatrix}\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{Bmatrix} \quad (3.97)$$

where,

$$S_{11}=S_{22}=\frac{1}{E} \quad (3.98)$$

$$S_{12}=-\frac{\nu}{E} \quad (3.99)$$

$$S_{66}=\frac{1}{G} \quad (3.100)$$

Lecture 27

Stress-strain relations for lamina with arbitrary orientation:

Consider an orthotropic lamina with its principal material axes oriented at an angle θ with the reference coordinate axes as shown in figure. Stresses and strains can be easily transformed from one set of axes to another.

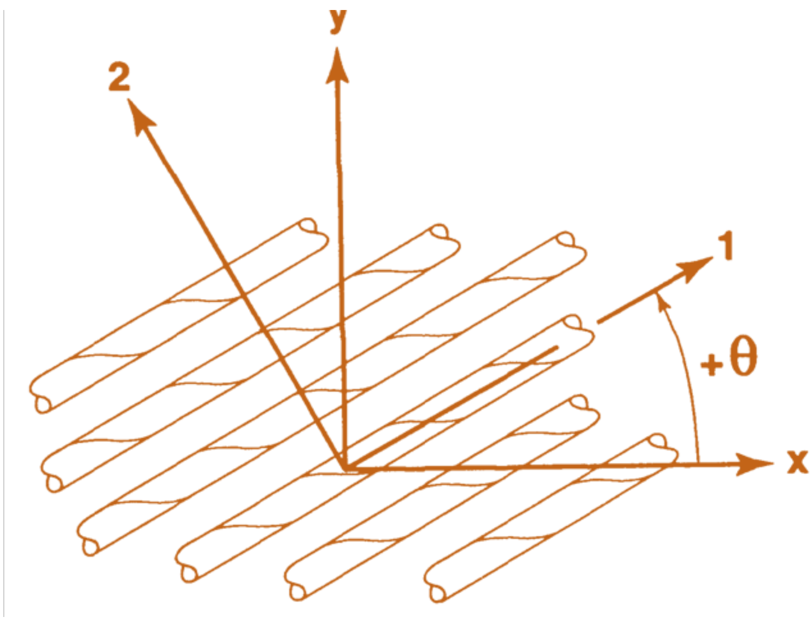


Fig. 3.13 lamina with arbitrary orientation

From elementary mechanics of materials the transformation equations for expressing stresses in a 1-2 coordinate system in terms of stresses in a x-y coordinate system,

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = [T] \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \tag{3.101}$$

$$[T] = \begin{bmatrix} cos^2\theta & sin^2\theta & 2sin\theta cos\theta \\ sin^2\theta & cos^2\theta & -2sin\theta cos\theta \\ -sin\theta cos\theta & sin\theta cos\theta & cos^2\theta - sin^2\theta \end{bmatrix} \tag{3.102}$$

The transformations are commonly written as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [T]^{-1} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} \tag{3.103}$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \frac{\gamma_{xy}}{2} \end{Bmatrix} = [T]^{-1} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \frac{\gamma_{12}}{2} \end{Bmatrix} \tag{3.104}$$

where, the transformation matrix $[T]^{-1}$ is given by

$$[T]^{-1} = \begin{bmatrix} cos^2\theta & sin^2\theta & -2sin\theta cos\theta \\ sin^2\theta & cos^2\theta & 2sin\theta cos\theta \\ sin\theta cos\theta & -sin\theta cos\theta & cos^2\theta - sin^2\theta \end{bmatrix} \tag{3.105}$$

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = [R] \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \frac{\gamma_{12}}{2} \end{Bmatrix} \tag{3.106}$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = [R] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \frac{\gamma_{xy}}{2} \end{Bmatrix} \tag{3.107}$$

where, $[R]$ is the Reuter matrix and is defined as

$$[R] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \tag{3.108}$$

For a specially orthotropic lamina whose principal material axes are aligned with the natural body axes,

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = [Q] \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} \tag{3.109}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [T]^{-1} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = [T]^{-1}[Q] \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} \tag{3.110}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [T]^{-1}[Q][R]\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \frac{\gamma_{12}}{2} \end{Bmatrix} \quad (3.111)$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [T]^{-1}[Q][R][T]\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \frac{\gamma_{xy}}{2} \end{Bmatrix} \quad (3.112)$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [T]^{-1}[Q][R][T][R]^{-1}\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad (3.113)$$

However, $[R][T][R]^{-1} = [T]^{-T}$ (3.114)

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [T]^{-1}[Q][T]^{-T}\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad (3.115)$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [\bar{Q}]\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad (3.116)$$

where, $[\bar{Q}]$ is the transformed reduced stiffness matrix.

$$[\bar{Q}] = [\blacksquare(T)]^{\uparrow}(-1)\ [Q]\ [\blacksquare(T)]^{\uparrow}(-T) \quad (3.117)$$

Thus, the stress-strain relations in x-y coordinates are

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad (3.118)$$

where,

$$\overline{Q}_{11} = Q_{11}\cos^4\theta + Q_{22}\sin^4\theta + 2(Q_{12}+2Q_{66})\sin^2\theta\cos^2\theta$$

$$\overline{Q}_{22} = Q_{11}\sin^4\theta + Q_{22}\cos^4\theta + 2(Q_{12}+2Q_{66})\sin^2\theta\cos^2\theta$$

$$\overline{Q_{12}} = (Q_{11}+Q_{22}-4Q_{66})\sin^2\theta\cos^2\theta + Q_{12}(\sin^4\theta+\cos^4\theta)$$

$$\overline{Q_{16}} = (Q_{11}-Q_{12}-2Q_{66})\sin\theta\cos^3\theta - (Q_{22}-Q_{12}-2Q_{66})\sin^3\theta\cos\theta$$

$$\overline{Q_{26}} = (Q_{11}-Q_{12}-2Q_{66})\sin^3\theta\cos\theta - (Q_{22}-Q_{12}-2Q_{66})\sin\theta\cos^3\theta$$

$$\overline{Q_{66}} = (Q_{11}+Q_{22}-2Q_{12}-2Q_{66})\sin^2\theta\cos^2\theta + Q_{66}(\sin^4\theta+\cos^4\theta)$$

(3.119)

If $\theta = 0^o$,

$$\bar{Q}_{11} = Q_{11}$$
$$\bar{Q}_{22} = Q_{22}$$
$$\bar{Q}_{12} = Q_{12}$$
$$\bar{Q}_{16} = \bar{Q}_{26} = 0$$
$$\bar{Q}_{66} = Q_{66} \qquad (3.120)$$

If $\theta = 90^o$,

$$\bar{Q}_{11} = Q_{22}$$
$$\bar{Q}22 = Q_{11}$$
$$\bar{Q}_{12} = Q_{12}$$
$$\bar{Q}_{16} = \bar{Q}_{26} = 0$$
$$\bar{Q}_{66} = Q_{66} \qquad (3.121)$$

Similarly, the strain-stress relation along the principal axes for a specially orthotropic lamina can be written as :

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} \qquad (3.122)$$

Therefore, the strain-stress relation along any arbitrary direction is given by

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = [\hat{S}] \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \qquad (3.123)$$

where, $\hat{S}$ is the transformed reduced compliance matrix

$$\hat{S} = [T]^T[S][T] \quad \text{and}$$
$$[T]^T = [R][T]^{-1}[R]^{-1} \qquad (3.124)$$

Thus, the strain-stress relations in x-y coordinates are

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \hat{S}_{11} & \hat{S}_{12} & \hat{S}_{16} \\ \hat{S}_{12} & \hat{S}_{22} & \hat{S}_{26} \\ \hat{S}_{16} & \hat{S}_{26} & \hat{S}_{66} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \qquad (3.125)$$

where,

$$\overline{S_{11}} = S_{11}\cos^4\theta + S_{22}\sin^4\theta + 2(S_{12} + 2S_{66})\sin^2\theta\cos^2\theta$$

$$\overline{S_{22}} = S_{11}\sin^4\theta + S_{22}\cos^4\theta + 2(S_{12} + 2S_{66})\sin^2\theta\cos^2\theta$$

$$\overline{S_{12}} = (S_{11} + S_{22} - S_{66})\sin^2\theta\cos^2\theta + S_{12}(\sin^4\theta + \cos^4\theta)$$

$$\overline{S_{16}} = (2S_{11} - 2S_{12} - S_{66})\sin\theta\cos^2\theta - (2S_{22} - 2S_{12} - S_{66})\sin^2\theta\cos\theta$$

$$\overline{S_{26}} = (2S_{11} - 2S_{12} - SQ_{66})\sin^2\theta\cos\theta - (2S_{22} - 2S_{12} - S_{66})\sin\theta\cos^2\theta$$

$$\overline{S_{66}} = 2(2S_{11} + 2S_{22} - 4S_{12} - S_{66})\sin^2\theta\cos^2\theta - S_{66}(\sin^4\theta + \cos^4\theta)$$

(3.126)

If θ = 0°,

$$\hat{S}_{11} = S_{11}\ ;\ \hat{S}_{22} = S_{22}\ ;\ \hat{S}_{12} = S_{12}\ ;\ \hat{S}_{16} = \hat{S}_{26} = 0\ ;\ \hat{S}_{66} = S_{66}$$

(3.127)

If θ = 90°,

$$\hat{S}_{11} = S_{22}\ ;\ \hat{S}_{22} = S_{11}\ ;\ \hat{S}_{12} = S_{12}\ ;\ \hat{S}_{16} = \hat{S}_{26} = 0\ ;\ \hat{S}_{66} = S_{66}$$

(3.128)

Lecture 28

Mechanical properties of Composites:

The mechanical properties of composite materials are determined by conducting standard tests framed by the American Society for Testing and Materials (ASTM). The followings are the list of ASTM standards, which are used to find the mechanical properties of composite materials.

Tensile test	- ASTM D3039
Compression test	- ASTM D3410
Flexural test	- ASTM D790
±45 Shear test	- ASTM D3518
In-plane Shear test	- ASTM D4255
Inter-laminar shear strength	- ASTM D2344

Tensile Strength test:

The tensile properties of composite laminate are determined in accordance with ASTM D3039. The tensile specimen is straight-sided and has the cross-section as shown in the figure.

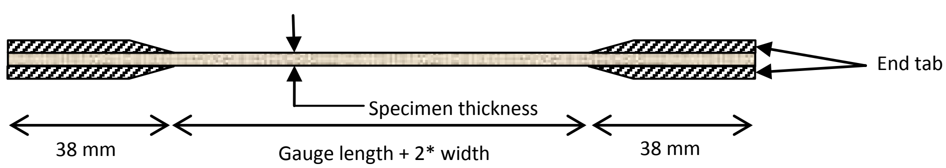


Figure 3.14 tensile test specimen

The tensile specimen is held in the testing machine by wedge action grips. Longitudinal and transverse strains are measured using strain gauges. Longitudinal tensile modulus E_{11} and the major Poisson's ratio ν_{12} are determined from the tension test data of 0^o unidirectional laminates. The transverse modulus E_{22} and the minor Poisson's ratio ν_{21} are determined from the tension test data of 90^o unidirectional laminates.

Figure 3.15 Tensile test apparatus

For an off-axis unidirectional specimen ($0^o < \theta < 90^o$), a tensile load creates both extension and shear deformations (since A_{16} and A_{26} are not equal to zero). Hence, the experimentally determined modulus of an off-axis specimen is corrected to obtain its true modulus using the equation:

$$E_{true} = (1-\eta)\, E_{experimental} \qquad (3.129)$$

where, $\eta = 3*S_{16}^2 / (S_{11}^2 [3*(S_{66}/S_{11}) + 2*(L/w)^2])$ (3.130)

L - the specimen length between grips

w - the specimen width

S_{ij} - the elements in the compliance matrix

Flexural Strength test:

Flexural strength and modulus are determined by ASTM test method D790. In this test, a rectangular cross section of composite beam speciment is loaded in a three-point bending mode.

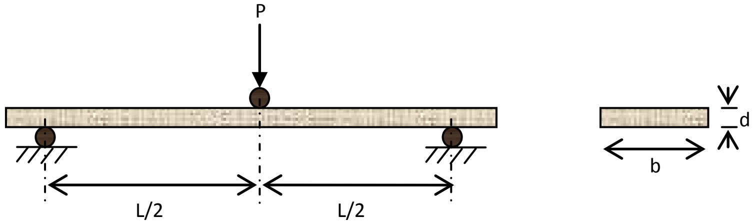


Figure 3.16 Flexural test setup

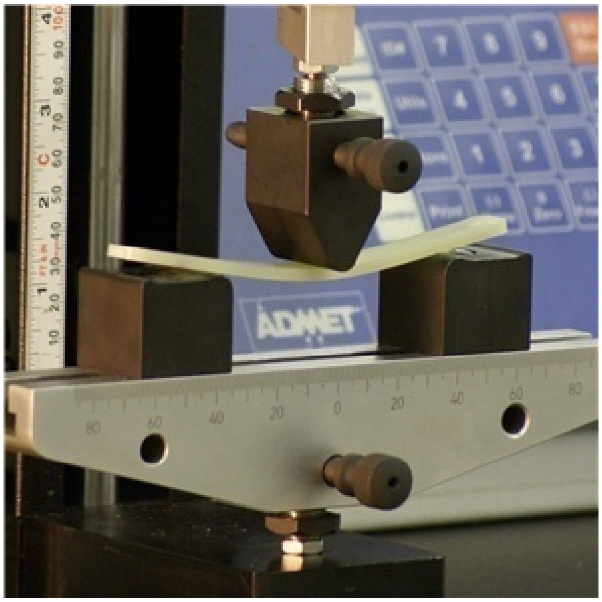


Figure 3.17 Flexural test apparatus - Three point loading

The maximum fiber stress at failure on the tension side of a flexural specimen is considered the flexural strength of the material. Therefore, the flexural strength is given by :

$$\sigma = \frac{M}{I} y \qquad (3.131)$$

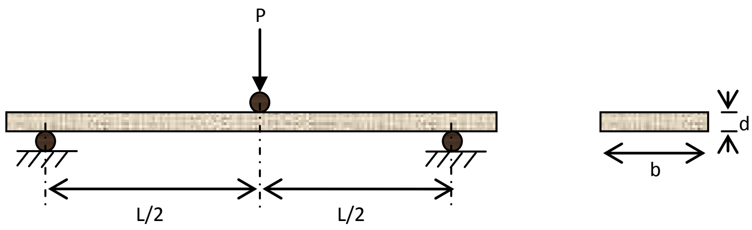
$$\sigma = \frac{\left(\frac{P}{2}\right)\left(\frac{l}{2}\right)\left(\frac{d}{2}\right)}{\frac{bd^3}{12}} \qquad (3.132)$$

$$\sigma = \frac{3Pl}{2bd^2} \qquad (3.133)$$

Inter-laminar Shear Strength (ILSS):

Inter-laminar shear strength of composite is the shear strength parallel to the plane of laminate. It is determined in accordance with ASTM D2344 using a short-beam shear test. It is applicable to all types of parallel fiber reinforced plastics and composites.

The thickness and width of the test specimen are measured before conditioning. The specimen is placed on a horizontal shear test fixture so that the fibers are parallel to the loading nose. The loading nose is then used to flex the specimen at a speed of .05 inches per minute until breakage. The force is then recorded. To determine shear strength, calculations are performed as given below.



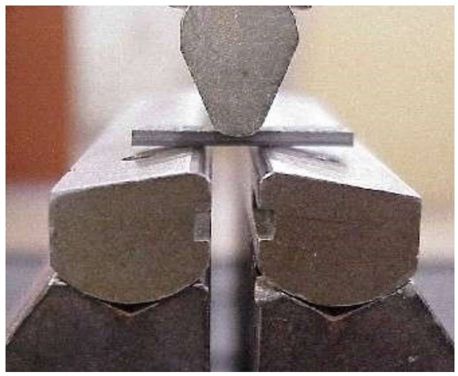


Figure 3.18 Short beam shear test

$$shear\ strength, \tau = \frac{V}{I}\int y\, ds \quad (3.134)$$

$$\tau = \frac{V}{I}\left(\frac{y^2}{2}\right)_0^{\frac{d}{2}} \quad (3.135)$$

$$\tau = \frac{Pd^2}{16\left(\frac{bd^3}{12}\right)} \quad (3.136)$$

$$\tau = \frac{3P}{4bd} \quad (3.137)$$

In-Plane Shear Strength (IPSS):

The shear modulus G_{12} and the ultimate shear strength τ_{12U} of unidirectional fiber-reinforced composites may be determined by any one of the following test procedures.

(i) ± 45^o shear test (ASTM D3518)

(ii) Iosipescu or V-notched shear test (ASTM D5379)

(iii) Two-rail or three-rail shear test (ASTM D4255)

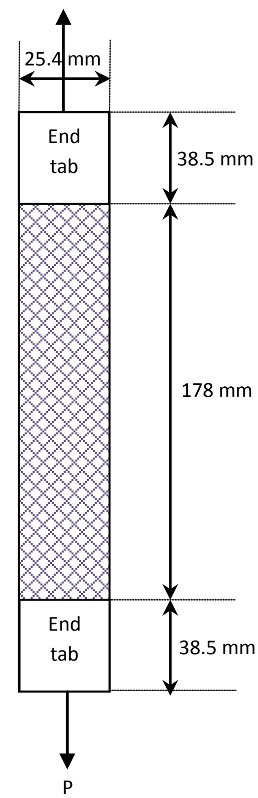


Figure 3.19 ± 45^{o} shear test configuration

In ± 45^{o} shear test, test specimens are placed in the grips of a universal tester at a specified grip separation and pulled until failure. Optional tabs can be bonded to the ends of the specimen to prevent gripping damage. The expressions to find the shear modulus and the shear strength is given by the expressions

$$G_{12} = \frac{\sigma_{xx}}{2\left(\varepsilon_{xx} - \varepsilon_{yy}\right)} \quad (3.138)$$

The shear strength is given by the expression

$$S_{12} = \frac{P_{\max}}{2bh} \quad (3.139)$$

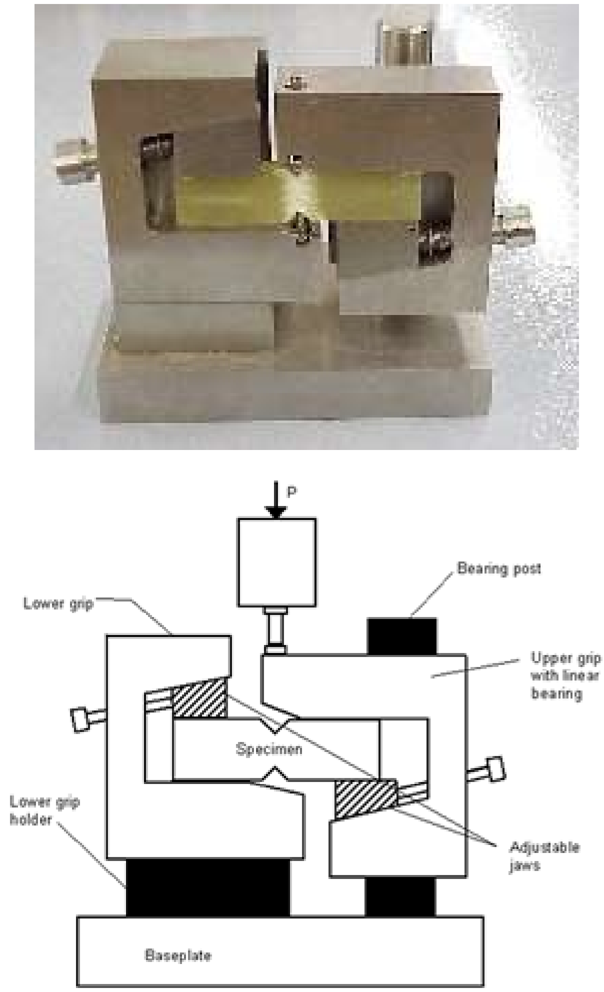


Figure 3.20 V-notched shear test

The Iosipescu or V-notched shear test uses a rectangular beam with symmetrical centrally located V-notches. The beam is loaded by a special fixture applying a shear loading at the V notch. Either in-plane or out-of-plane shear properties may be evaluated, depending upon the orientation of the material co-ordinate system relative to the loading axis. The notched specimen is loaded by introducing a relative displacement between two halves of the test fixture. The shear stress is calculated as

$$\tau = \frac{P}{wh} \qquad (3.140)$$

where P - applied load

w - distance between the notches

h - specimen thickness

Figure 3.21 Two-rail shear test

In two-rail shear test, two pairs of steel rails are fastened along the long edges of a 76.2 mm wide and 152.4 mm long rectangular specimen, usually by three bolts on each side. At the other two edges, the specimen remains free. A tensile load is applied to the rails. That will induce an in-plane shear load on the tested laminate. The shear strength is calculated by using the expression

$$\tau = \frac{P}{Lh} \tag{3.141}$$

where, L - the specimen length

h - the specimen thickness